⑥ The Wigner - Eckart Theorem.

→ Matrix elements of Tensor operators.

$$\langle \alpha' j' m' | T_q^{(k)} | \alpha j m \rangle = \underbrace{\langle j k ; m q | j k ; j' m' \rangle}_{①} \frac{\overbrace{\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle}^{②}}{\sqrt{2j+1}}$$

① $\langle j k ; m q | j k ; j' m' \rangle \equiv C^{jk}_{mq;j'm'}$ ... "Selection rule"

$= 0$ unless $m' = q + m$ and $|j-k| \leq j' \leq j+k$

② $\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle$: "reduced matrix element"

independent of m and m'.

proof. use $[J_\pm, T_q^{(k)}] = \hbar \sqrt{(k \mp q)(k \pm q + 1)}\, T^{(k)}_{q\pm 1}$.

$$\Rightarrow \langle \alpha' j' m' | \underbrace{[J_\pm, T_q^{(k)}]}_{J_\pm T_q^{(k)} - T_q^{(k)} J_\pm} | \alpha j m \rangle = \hbar \sqrt{(k \mp q)(k \pm q + 1)} \langle \alpha' j' m' | T^{(k)}_{q\pm 1} | \alpha j m \rangle$$

$$\sqrt{(j' \pm m')(j' \mp m' + 1)} \langle \alpha', j', m' \mp 1 | T_q^{(k)} | \alpha j m \rangle$$

$$= \sqrt{(j \mp m)(j \pm m + 1)} \langle \alpha' j' m' | T_q^{(k)} | \alpha, j, m \pm 1 \rangle$$

$$+ \sqrt{(k \mp q)(k \pm q + 1)} \langle \alpha' j' m' | T^{(k)}_{q \pm 1} | \alpha j m \rangle .$$

comparing with the recursion relation of the CG coeffs :

$$\sqrt{(j \pm m)(j \mp m + 1)}\, C^{j_1 j_2}_{m_1, m_2; j, m \mp 1}$$

$$= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\, C^{j_1 j_2}_{m_1 \pm 1, m_2; j m}$$

$$+ \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\, C^{j_1 j_2}_{m_1, m_2 \pm 1; j m}$$

NOTE :

what we had : $\pm$

↓

← here : $\mp$

← Two recursion relations become identical

when we put $j' \to j$, $m' \to m$, $j \to j_1$, $m \to m_1$,

$k \to j_2$, $q \to m_2$. for a given $(j_1 j_2 j)$

Thus, $\langle \alpha' j' m' | T_q^{(k)} | \alpha j m \rangle$

$$\propto C^{jk}_{mq;j'm'} = \langle jk; mq | jk; j'm' \rangle$$ #

⑰ Consequences of the Wigner-Eckart Theorem.

a. magnetic moment $\vec{\mu}$ (a first-rank tensor)

$$\langle jm | \mu_z | jm \rangle = C^{j1}_{m0;jm} \frac{\langle j \| \mu_z \| j \rangle}{\sqrt{2j+1}}$$

μ (what we find in the table) $\quad \| \quad \mu_z \Rightarrow T_0^{(1)}$

$$\equiv \langle jj | \mu_z | jj \rangle .$$

Using $C^{j1}_{m0;jm} = \dfrac{m}{\sqrt{j(j+1)}}$, (show it by yourself)

$$\mu = \sqrt{\frac{j}{j+1}} \frac{\langle j \| \mu_z \| j \rangle}{\sqrt{2j+1}} \to \langle j \| \mu_z \| j \rangle = \sqrt{\frac{(j+1)(2j+1)}{j}} \mu .$$

$$\therefore \langle jm | \mu_z | jm \rangle = \frac{m}{\sqrt{j(j+1)}} \cdot \frac{1}{\sqrt{2j+1}} \cdot \sqrt{\frac{(j+1)(2j+1)}{j}} \mu$$

$$= \frac{m}{j} \mu$$

b. projection Theorem and Landé g-factor.

The projection Theorem

$$\langle \alpha' j m' | T^{(1)}_q | \alpha j m \rangle = \frac{\langle \alpha' j \cancel{m} | \vec{J}\cdot\vec{T}^{(1)} | \alpha j \cancel{m} \rangle}{\hbar^2 j(j+1)} \cdot \langle j m' | J_q | j m \rangle$$

where J_q : $J_{\pm 1} = \mp \frac{1}{\sqrt{2}} (J_x \pm i J_y) = \mp J_{\pm} \cdot \frac{1}{\sqrt{2}}$

$J_0 = J_z$.

proof.

$$\langle \alpha' j m | \vec{J}\cdot\vec{T}^{(1)} | \alpha j m \rangle = \langle \alpha' j m | \left(J_0 T^{(1)}_0 - J_{+1} T^{(1)}_{-1} - J_{-1} T^{(1)}_{+1} \right) | \alpha . j m \rangle$$

$$= m\hbar \langle \alpha', j m | T^{(1)}_0 | \alpha j m \rangle$$
$$+ \frac{\hbar}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \langle \alpha', j . m-1 | T^{(1)}_{-1} | \alpha j m \rangle$$
$$- \frac{\hbar}{\sqrt{2}} \sqrt{(j-m)(j+m+1)} \langle \alpha', j . m+1 | T^{(1)}_{+1} | \alpha . j m \rangle$$

also, by using the Wigner-Eckart theorem,

$$= C_{j\cancel{m}} \langle \alpha' j \| \vec{T}^{(1)} \| \alpha j \rangle$$

$|j,m\rangle \Leftarrow |j,m\rangle \otimes |0,0\rangle$

$\uparrow$ $\vec{J}\cdot\vec{T}^{(1)}$ is a scalar operator. — indep. of m

$$\Rightarrow \langle \alpha' j \cancel{m} | \vec{J}\cdot\vec{T}^{(1)} | \alpha j \cancel{m} \rangle = C_j \langle \alpha' j \| \vec{T}^{(1)} \| \alpha j \rangle .$$

If we let $\vec{T}^{(1)} = \vec{J}$, $\alpha' \to \alpha$, then

$$\langle \cancel{\alpha} j m | \vec{J}^2 | \cancel{\alpha} j m \rangle = C_j \langle \cancel{\alpha'} j \| \vec{J} \| \cancel{\alpha} . j \rangle = \hbar^2 j(j+1) .$$

also, the Wigner-Eckart theorem says.

$$\frac{\langle \alpha' j m' | T^{(1)}_q | \alpha , j m \rangle}{\langle \cancel{\alpha} , j m' | J_q | \cancel{\alpha} , j m \rangle} = \frac{\langle \alpha' j \| \vec{T}^{(1)} \| \alpha j \rangle}{\langle \cancel{\alpha} j \| \vec{J} \| \cancel{\alpha} j \rangle}$$

$\|$ α has no effects on $\vec{J}$.

$$\Rightarrow \frac{\langle \alpha' j \cancel{m} | \vec{J}\cdot\vec{T}^{(1)} | \alpha j \cancel{m} \rangle}{\hbar^2 j(j+1)} = \frac{\langle \alpha' j m' | T^{(1)}_q | \alpha j m \rangle}{\langle \cancel{\alpha} j m' | J_q | \cancel{\alpha} j m \rangle}$$

#

• The Landé g-factor

The total magnetic moment of an electron is

$$\vec{\mu} = \vec{\mu}_S + \vec{\mu}_L = -\mu_B (g_s \vec{S} + g_\ell \vec{L}) \, .$$

$$= - g_J \mu_B \vec{J} \qquad \parallel \text{ Total ang. mom. } \vec{J} = \vec{L} + \vec{S} \, .$$

We can compute g_J : the Landé-g factor.

by using the projection theorem :

$$\langle jm | \mu_z | jm \rangle = \frac{\langle j | \vec{J} \cdot \vec{\mu} | j \rangle}{j \cdot (j+1) \hbar^2} \cdot \langle jm | J_z | jm \rangle$$

$$= \frac{m}{j(j+1)\hbar} \langle j | \vec{J} \cdot \vec{\mu} | j \rangle$$

This can be rewritten in terms of $\vec{L}$ and $\vec{S}$ as

$$= - \frac{m}{j(j+1)\hbar} \mu_B \langle j | (\vec{L} + \vec{S}) \cdot (g_\ell \vec{L} + g_s \vec{S}) | j \rangle \, .$$

Now, using

$$\langle j | \vec{L}^2 | j \rangle = \ell (\ell+1) \hbar^2$$

$$\langle j | \vec{S}^2 | j \rangle = s (s+1) \hbar^2$$

$$\langle j | \vec{L} \cdot \vec{S} | j \rangle = \frac{1}{2} \langle j | \vec{J}^2 - \vec{L}^2 - \vec{S}^2 | j \rangle$$

$$= \frac{1}{2} \left[j(j+1) - \ell(\ell+1) - s(s+1) \right] \hbar^2$$

$$\Rightarrow \langle jm | \mu_z | jm \rangle = - \frac{m \mu_B \hbar}{j(j+1)} \left[g_\ell \frac{j(j+1) + \ell(\ell+1) - s(s+1)}{2} + g_s \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2} \right]$$

$$= - g_J \mu_B m \hbar$$

For $g_\ell = 1$, $g_s = 2$ for an electron,

$$g_J = 1 + \frac{1}{2j(j+1)} \left[j(j+1) - \ell(\ell+1) + s(s+1) \right]$$ //

c. selection rule

$$\langle j'm' | T^{(k)}_{q} | jm \rangle \propto C^{j\,k}_{m\,q;j'm'}$$

$\neq 0$ only when $m' = m+q$
$j' = j+k, \cdots, |j-k|$.

– a spin-j tensor observable in a "unpolarized" state

$$\langle T^{(j)}_m \rangle = \frac{1}{2j'+1} \sum_{m'=-j'}^{j'} \langle j'm' | T^{(j)}_m | j'm' \rangle$$

Survives only when $j=0$, $m=0$!

(The sum vanishes, otherwise)

$$= \delta_{j0}\, \delta_{m0} \frac{\langle j' \| T^{(0)} \| j' \rangle}{\sqrt{2j'+1}}$$

→ It vanishes unless it's a scalar operator,
in a unpolarized state.

(electric dipole transition)

– Stark effect : you need to compute

$\langle n' \ell' m' | z | z, \ell, m \rangle$!

→ 0 unless $\ell' = \ell \pm 1$, $m' = m$ because $z = T^{(1)}_0$.

In general , $\langle n'\ell'm' | \vec{r} | n \ell m \rangle \neq 0$ when

$\Delta m = 0$ if $\vec{r} = \hat{z}$

$\Delta m = \pm 1$ if $\vec{r} = \hat{x}$ or $\hat{y}$.

- Emission and Absorption of radiation

Transition probability $\propto |M|^2$ || * The Fermi Golden Rule

$$M = \langle \underset{100}{n'l'm'}; \gamma | T | \underset{21m}{nlm} \rangle$$: For emission of the photon from 2p to 1s.

In general, $T \equiv \sum_{k=1}^{\infty} \vec{S}_k \cdot \vec{K}_k = \sum_{k=1}^{\infty} \sum_{q=-k}^{k} (-1)^q S_q^{(k)} K_{-q}^{(k)}$.

atom ($\vec{S}_k$), radiation ($\vec{K}_k$)

in terms of the irreducible spherical tensors.

$$M = \sum_{k=0}^{\infty} \sum_{q=-k}^{k} (-1)^q \langle 100 | S_q^{(k)} | 21m \rangle_a \langle \gamma | K_{-q}^{(k)} | 0 \rangle_r$$

(atom) (radiation)

The Wigner-Eckart Theorem says,

$$\langle 100 | S_q^{(k)} | 21m \rangle = \underline{\underline{C_{m\,q;\,00}^{1\,k}}} \frac{\langle 10 \| S^{(k)} \| 21 \rangle}{\sqrt{3}}$$

Vanishes unless $k=1$ and $m=-q$.

Thus. $M = (-1)^{-m} \langle 100 | S_{-m}^{(1)} | 21m \rangle_a \langle \gamma | K_m^{(1)} | 0 \rangle_r$

: only one term survives!

: The photon carries angular momentum 1 !